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Symmetries and Conservation Laws for Generalized Hamiltonian Systems

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A class of dynamical systems which locally correspond to a general first-order system of Euler–Lagrange equations is studied on a contact manifold. These systems, called self-adjoint, can be regarded as generalizations of (time-dependent) Hamiltonian systems. It is shown that each one-parameter family of symmetries of the underlying contact form defines a parameter-dependent constant of the motion and vice versa. Next, an extension of the classical concept of canonical transformations is introduced. One-parameter families of canonical transformations are studied and shown to be generated as solutions of a self-adjoint system. Some of the results are illustrated on the Emden equation.

1. INTRODUCTION

In recent papers (Sarlet and Cantrijn, 1978a, b), we have discussed several aspects of general systems of first-order ordinary differential equations, which are derivable from a fixed-endpoint variational principle, and are regarded as generalizations of Hamilton's equations. These systems were called self-adjoint (SA), the corresponding linear variational form being self-adjoint in the sense of the calculus of variations (see, e.g., Santilli, 1978a).

We recall that a general SA system in $2n$ dimensions is given by (see Sarlet and Cantrijn, 1978a)

$$C_{ij}(t,x)\dot{x}^j + D_i(t,x) = 0, \qquad i = 1,\ldots,2n \tag{1}$$

where all coefficients are assumed to be of class C^∞ in some open domain of

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$\mathbb{R}^{2n+1}$, (C_{ij}) is supposed to be regular (at least locally), and the following conditions are satisfied:

$$C_{ij} = -C_{ji} \tag{2a}$$

$$\frac{\partial C_{ij}}{\partial x^k} + \frac{\partial C_{ki}}{\partial x^j} + \frac{\partial C_{jk}}{\partial x^i} = 0 \tag{2b}$$

$$\frac{\partial C_{ij}}{\partial t} = \frac{\partial D_i}{\partial x^j} - \frac{\partial D_j}{\partial x^i} \tag{2c}$$

The distinction between conditions (2b) and (2c) is merely a consequence of the fact that time is not treated on the same footing as the other coordinates. However, when considering t as a supplementary coordinate, (2b) and (2c) simply express that the 2-form

$$\sum_{i<j} C_{ij}\, dx^i \wedge dx^j - D_i\, dt \wedge dx^i \tag{3}$$

is closed in $\mathbb{R}^{2n+1}$. According to the Poincaré lemma this 2-form will then be locally exact, which guarantees the existence of C^∞ functions R_i ($i = 1,\ldots,2n$) and H such that

$$C_{ij} = \frac{\partial R_j}{\partial x^i} - \frac{\partial R_i}{\partial x^j}, \qquad D_i = -\frac{\partial H}{\partial x^i} - \frac{\partial R_i}{\partial t} \tag{4}$$

General SA systems (1), (2) constitute an extension of the classical Hamilton equations, which are easily recovered when (C_{ij}) is taken to be the canonical symplectic matrix. Note that in Sarlet and Cantrijn (1978a), following the spirit of classical nonrelativistic mechanics, t was treated purely as a parameter in conditions (2). The relations (4) therefore were derived from a parametric version of the Poincaré lemma. In the same spirit, coordinate transformations were considered to be time-preserving. In thinking of the analog of classical canonical transformations, it then seemed natural (in this local context) to study regular, time-preserving transformations

$$(t, x) \to (t, y(t, x))$$

which leave the 2-form (3) invariant up to terms containing dt. Such transformations were called "identity-isotopic" (or briefly I_C), adopting in this way a terminology introduced by Santilli (1978b). The reason for using this peculiar denomination instead of the word canonical lies in the fact that solutions of a general SA system, considered as transformations from the initial state, fail to be I_C transformations, unless all C_{ij}'s are time independent. This clearly raises questions concerning the possible generalization of other familiar concepts in Hamiltonian mechanics. Indeed, it is precisely this property of classical canonical transformations which lies at the heart of the formulation of the Hamilton–Jacobi theory and the theory of canonical symmetries and related conservation laws for Hamiltonian systems.

Previously (Sarlet and Cantrijn 1978b) we have dealt with the generalized Hamilton–Jacobi problem, by enlarging the class of I_C transformations, still within the local context of the foregoing considerations.

In the present paper we will focus our attention on the study of one-parameter families of transformations, symmetries, and conservation laws for general SA systems. We will aim at a more global approach to the problem, and in doing so we will show that an appropriate definition for generalized canonical transformations will simply arise from relaxing slightly the previous requirement of time preservation.

We start by putting a contact structure on the trivial bundle $\mathbb{R}\times M$, M being an even-dimensional manifold and $\mathbb{R}$ the set of reals, allowing in this way an explicit mentioning of time as globally defined variable. A class of vector fields is defined (Section 2), corresponding locally to general SA systems of type (1), (4). For consistency in terminology, such vector fields will also be called "self-adjoint" (SA). They contain the usual definitions of Lagrangian systems on a tangent bundle and Hamiltonian systems on a cotangent bundle as special cases.

In Section 3 some general aspects of one-parameter families of diffeomorphisms are recalled. A notion of symmetry for SA-vector fields is introduced in Section 4, which is general enough to allow for arbitrary time variations, and is related to the existence of a first integral.

In Section 5 we seek for a suitable definition of canonical transformations, which stays as closely as possible to the spirit of the classical definition. We show in Section 6 that one-parameter families of canonical transformations are themselves generated by a SA-vector field, and we continue (Section 7) with the special case of canonical symmetries. The paper ends with an illustration and a discussion of the results.

Although some of the material presented in this paper might also be derived from more abstract settings[2], it is our feeling that even in modern treatments the case of time-dependent Hamiltonian mechanics governed by

[2]We here think for instance of the so-called momentum-map theory, which can be found, e.g., in Abraham and Marsden (1978).

a general contact form (i.e., not the pull-back of a time-independent symplectic form), or in other words the peculiarities of allowing an explicit time dependence of the matrix (C_{ij}) in the local description (1), are generally overlooked.

On the one hand, we aim in our approach at taking full advantage of the conciseness provided by the use of modern differential geometrical concepts. On the other hand, we want to keep the treatment readable by nonspecialists in the field. This means that we will only appeal to the most basic operations on vector fields and differential forms, and we will give a local representation of all results in which the classical treatment can be recovered. Perhaps the general framework of this paper is most closely related to a recent contribution by Crampin (1977), the main difference being that Crampin's study is Lagrangian in character, while the spirit of the present paper is more Hamiltonian.

The notations adopted in this paper are the following: the sets of vector fields, p-forms, and diffeomorphisms (all of class C^∞) on a differentiable manifold N are, respectively, denoted by $\mathfrak{X}(N)$, $\Omega^p(N)$, and Diff(N). The set of real-valued C^∞ functions on N is represented by $C^\infty(N)$. The contraction of a vector field X and a general p-form α is denoted by $i_X\alpha$, whereas, in case α is a 1-form, the notation $\langle X, \alpha\rangle$ is also frequently used.

2. SELF-ADJOINT VECTOR FIELDS

Throughout this paper we will be working on the trivial bundle $\mathbb{R}\times M$ over the real line, with M a $2n$-dimensional real C^∞-differentiable manifold. Henceforth, t will refer both to the time variable and the projection operator from $\mathbb{R}\times M$ onto $\mathbb{R}$.

Suppose we are given a 1-form $\theta\in\Omega^1(\mathbb{R}\times M)$ for which $d\theta$ is of constant rank $2n$. The 2-form $d\theta$ then defines an exact contact structure on $\mathbb{R}\times M$. Its characteristic bundle, denoted by $\mathcal{C}(d\theta)$, is one-dimensional.

In addition, we require that for some characteristic vector field Y of $d\theta$ the coupling with dt must be nowhere vanishing (i.e., $\langle Y, dt\rangle\neq 0$). On the contact manifold $(\mathbb{R}\times M, d\theta)$ we then introduce the following concept.

Definition 2.1. A vector field X on $\mathbb{R}\times M$ is called self-adjoint (SA) with respect to the given contact structure $d\theta$ iff

(i) $i_X d\theta=0$ (5a)

(ii) $\langle X, dt\rangle=1$ (5b)

The first of these conditions states that X must be a characteristic vector field of $d\theta$. Since $\mathcal{C}(d\theta)$ is one-dimensional and taking into account the above assumption made on dt, we immediately obtain the following corollary.

Corollary 2.1. Conditions (5a) and (5b) define a unique vector field on $\mathbb{R}\times M$. ■

One also easily verifies that the original 1-form θ may be replaced in the above definition by any 1-form θ' for which $d\theta=d\theta'$, i.e., which locally differs from θ by at most a total differential.

Let us now first check that this definition locally gives rise to a self-adjoint system of differential equations of type (1). Later on, for comparison with the description of SA systems given in Sarlet and Cantrijn (1978a), we will be more specific about the choice of a model for local representation of our results. For the time being it suffices to say that in any local coordinate system $(t, x^1,\ldots, x^{2n})$ on $\mathbb{R}\times M$, writing θ as

$$\theta=R_i\,dx^i-H\,dt$$

we get

$$d\theta=\sum_{i<j} C_{ij}\,dx^i\wedge dx^j-D_i\,dt\wedge dx^i$$

with C_{ij} and D_i as in (4). In view of the assumptions made on $d\theta$ and dt, the matrix (C_{ij}) must be regular. If the self-adjoint vector field with respect to $d\theta$ is represented by $X=\xi^i\partial/\partial x^i+\eta\partial/\partial t$, we derive from (5) the following relations for its components:

$$\eta=1 \tag{6a}$$

$$C_{ij}\xi^j+D_i=0, \qquad i=1,\ldots,2n \tag{6b}$$

$$D_j\xi^j=0 \tag{6c}$$

(where summation always runs from 1 to $2n$). From (6b) we obtain $\xi^i=-C^{ij}D_j$, with $(C^{ij})=(C_{ij})^{-1}$. Herewith (6c) is seen to be identically satisfied. Hence, taking into account (6a) and replacing D_j by its explicit expression, the system of differential equations associated with X can be reduced to

$$\dot{x}^i=C^{ij}\left(\frac{\partial H}{\partial x^j}+\frac{\partial R_j}{\partial t}\right), \qquad i=1,\ldots,2n$$

where the dot indicates differentiation with respect to t. This is precisely the normal form of a general SA system (1). Before proceeding it is worth mentioning that the definition of SA vector field contains the following special cases:

(1) Suppose M is a symplectic manifold on which an exact symplectic form $\omega=d\rho$ is given (e.g., let M be the cotangent bundle of an n-dimensional manifold and ρ the canonical 1-form $p_i\, dq^i$). Let p_2: $\mathbb{R}\times M\rightarrow M$ denote the natural projection operator and define a 1-form θ on $\mathbb{R}\times M$ by $\theta=p_2^*\rho-H\,dt$ for some function $H\in C^\infty(\mathbb{R}\times M)$. The 2-form

$$d\theta=p_2^*\omega-dH\wedge dt$$

then satisfies all the required conditions and the SA vector field corresponding to this contact form is precisely the time-dependent Hamiltonian vector field with Hamiltonian H, as defined according to the Cartan point of view (see, e.g., Hermann, 1968).

(2) Let M be the tangent bundle TN of some n-dimensional manifold N and consider a regular (time-dependent) Lagrange function $L\in C^\infty(\mathbb{R}\times TN)$. With respect to a set of natural coordinates $(t, q^1,\ldots, q^n, \dot{q}^1,\ldots, \dot{q}^n)$ on $\mathbb{R}\times TN$, consider the so-called Cartan form θ associated with L,

$$\theta=L\,dt+\frac{\partial L}{\partial \dot{q}^i}\left(dq^i-\dot{q}^i\,dt\right)$$

The assumed regularity of L implies $d\theta$ to be of rank $2n$. With this particular choice for θ, equations (5) coincide with the definition of the Lagrangian vector field corresponding to L (see Crampin, 1977, or Sternberg, 1964, p. 159).

Before dealing with the study of symmetries for SA systems we first briefly recall in the next section some general properties concerning one-parameter families of diffeomorphisms on $\mathbb{R}\times M$.

3. ONE-PARAMETER FAMILIES OF TRANSFORMATIONS

Let $\mathcal{F}=\{F_\lambda\,|\,\lambda\in\mathbb{R}\}$ be a smooth one-parameter family of transformations from $\mathbb{R}\times M$ onto itself, i.e., (i) $F_\lambda\in \mathrm{Diff}(\mathbb{R}\times M)$; (ii) F_0 is the identity map on $\mathbb{R}\times M$; (iii) the map F: $\mathbb{R}\times\mathbb{R}\times M\rightarrow\mathbb{R}\times M$ with $F(\lambda, t, m)=F_\lambda(t, m)$ is of class C^∞.

Since no group structure has been assumed, $\mathcal{F}$ will not define a (unique) vector field on $\mathbb{R}\times M$. However, one can always associate to $\mathcal{F}$ a family of vector fields in a very precise way. To make this clear, the usual procedure consists in first passing from the given family $\mathcal{F}$ to a one-parameter group of transformations $\mathcal{G}=\{\tilde{G}_\mu\,|\,\mu\in\mathbb{R}\}$ on the extended bundle $\mathbb{R}\times(\mathbb{R}\times M)$, defined by

$$\tilde{G}_\mu(\lambda, t, m) = \left(\lambda+\mu, F_{\lambda+\mu}\left[F_\lambda^{-1}(t, m)\right]\right) \tag{7}$$

The $\tilde{G}_\mu$'s are clearly C^∞ diffeomorphisms forming a group under composition. Hence, they determine the flow of a vector field $\tilde{Z}$ on $\mathbb{R}\times(\mathbb{R}\times M)$, the so-called "infinitesimal generator" of $\mathcal{G}$. When projecting the integral curves of $\tilde{Z}$ onto the original space $\mathbb{R}\times M$ we recover the orbits defined by the given family $\mathcal{F}$. From (7) it is seen that $\tilde{Z}$ can be split up as follows:

$$\tilde{Z} = \frac{\partial}{\partial\lambda} + Z \tag{8}$$

with Z a λ-dependent vector field on $\mathbb{R}\times M$. For each fixed value of λ we further introduce the shorthand notation

$$Z|_{\{\lambda\}\times(\mathbb{R}\times M)} = Z_\lambda$$

Corresponding to $\mathcal{F}$ we have thus constructed in an unambiguous way a one-parameter family of vector fields Z_λ on $\mathbb{R}\times M$.

Conversely, assume we are given a smooth one-parameter family of vector fields $Z_\lambda \in \mathfrak{X}(\mathbb{R}\times M)$. The Z_λ's define a vector field $\tilde{Z}$ on $\mathbb{R}\times(\mathbb{R}\times M)$ according to (8), with $Z(\lambda, t, m) = Z_\lambda(t, m)$. In a neighborhood of each point of $\mathbb{R}\times(\mathbb{R}\times M)$, $\tilde{Z}$ induces a local one-parameter (pseudo-) group of diffeomorphisms $\{\tilde{G}_\mu | \mu \in I\}$, with I some open interval centered at the origin and where the $\tilde{G}_\mu$'s are of the form $\tilde{G}_\mu(\lambda, t, m) = (\lambda+\mu, G_\mu(\lambda, t, m))$ for some differentiable mappings G_μ. Putting

$$F_\mu(t, m) = G_\mu(0, t, m)$$

we obtain a local family of diffeomorphisms $\{F_\mu | \mu \in I\}$ on $\mathbb{R}\times M$ which, by construction, is completely determined by the given family of vector fields Z_λ.

To close this section we mention one of the basic formulas from the calculus of differential forms, which will be of great use later on. Consider again a smooth one-parameter family of transformations $\{F_\lambda | \lambda \in \mathbb{R}\}$ on $\mathbb{R}\times M$, together with a smooth family of p-forms $\{\alpha_\lambda | \lambda \in \mathbb{R}\}$. The following relation holds:

$$\frac{d}{d\lambda}(F_\lambda^*\alpha_\lambda) = F_\lambda^*\left(\frac{d\alpha_\lambda}{d\lambda}\right) + F_\lambda^*(i_{Z_\lambda} d\alpha_\lambda) + d\left[F_\lambda^*(i_{Z_\lambda}\alpha_\lambda)\right] \tag{9}$$

(Guillemin and Sternberg, 1977, p. 110).

4. SYMMETRIES AND RELATED CONSERVATION LAWS

Although it constitutes one of the major objectives in many treatments on dynamical systems, no precise and generally accepted definition of symmetry seems to exist. The way in which the concept of symmetry enters a certain theory often depends for instance on the specific nature of the systems it will be applied to, as well as on the general framework in which the analysis takes place (e.g., analytical or geometrical). In the case of a SA system it seems quite natural to introduce a notion of symmetry in terms of the contact form $d\theta$ which completely determines the structure of the corresponding vector field [up to some "normalization" imposed by (5b)]. We therefore propose the following definition.

Definition 4.1. A mapping $F \in \mathrm{Diff}(\mathbb{R} \times M)$ is called a symmetry of the contact form $d\theta$ iff

$$F_* d\theta = d\theta \tag{10}$$

Before investigating the influence of a symmetry of $d\theta$ on the corresponding SA vector field, it may be advisable to give a precise meaning to the expression "symmetry of a vector field." Henceforth we adopt the following definition.

Definition 4.2. A mapping $F \in \mathrm{Diff}(\mathbb{R} \times M)$ is called a symmetry of a vector field $Y \in \mathfrak{X}(\mathbb{R} \times M)$ iff

$$F_* Y = Y$$

According to this definition, a symmetry of a vector field transforms the set of integral curves of that vector field onto itself (without altering the parametrization of these curves). Let us now consider a symmetry F of $d\theta$. If F_* acts on (5a) we obtain, in view of (10),

$$i_{F_* X} d\theta = 0$$

Hence, $F_* X$ belongs to $\mathcal{C}(d\theta)$. The latter being one-dimensional and containing X, there must exist a function $f \in C^\infty(\mathbb{R} \times M)$ such that

$$F_* X = fX \tag{11}$$

Consequently, a symmetry of $d\theta$ will in general not be a symmetry of the SA vector field X (in the sense of Definition 4.2). From (11) we can only conclude that F permutes integral curves of X among themselves, allowing, however, a change of parametrization along these curves.

For the remainder of this section we will be dealing with one-parameter families of symmetries of $d\theta$.

For convenience we henceforth assume in this section that M (and thus also $\mathbb{R}\times M$) is simply connected. Let $\mathcal{F}=\{F_\lambda|\lambda\in\mathbb{R}\}$ be a smooth one-parameter family of transformations whereby each F_λ is a symmetry of $d\theta$, i.e.,

$$(F_\lambda)_* \, d\theta = d\theta \tag{12}$$

Since $\mathbb{R}\times M$ is simply connected, every closed 1-form is globally exact. Hence, (12) implies the existence of functions $S_\lambda \in C^\infty(\mathbb{R}\times M)$, smoothly depending on λ, such that

$$(F_\lambda)_*\theta = \theta + dS_\lambda$$

or equivalently

$$\theta = F_\lambda^*(\theta + dS_\lambda) \tag{13}$$

[If $\mathbb{R}\times M$ is not simply connected, (13) of course still holds locally.]

Proposition 4.1. Corresponding to each smooth family $\mathcal{F}=\{F_\lambda|\lambda\in\mathbb{R}\}$ of $d\theta$ symmetries there exists a λ-dependent first integral of the SA vector field X, which in terms of the S_λ is defined by

$$W_\lambda = \frac{dS_\lambda}{d\lambda} + L_{Z_\lambda} S_\lambda + i_{Z_\lambda}\theta \tag{14}$$

Proof. Applying (9) to (13) we obtain

$$0 = F_\lambda^*\left[\frac{d}{d\lambda}(dS_\lambda)\right] + F_\lambda^*\left(i_{Z_\lambda}\, d\theta\right) + d\left[F_\lambda^* i_{Z_\lambda}(\theta + dS_\lambda)\right]$$

with Z_λ defined as in Section 3. Rearranging terms we get

$$0 = F_\lambda^*\left(i_{Z_\lambda} d\theta\right) + F_\lambda^* d\left(\frac{dS_\lambda}{d\lambda} + L_{Z_\lambda} S_\lambda + i_{Z_\lambda}\theta\right)$$

where L_{Z_λ} is the Lie derivative with respect to Z_λ. Each F_λ being a diffeomorphism, it follows that

$$i_{Z_\lambda} d\theta = -dW_\lambda \tag{15}$$

with W_λ given by (14). The W_λ's are real-valued C^∞ functions which smoothly depend on λ. Using (15) and (5a) we get

$$L_X W_\lambda = i_X dW_\lambda = -i_X i_{Z_\lambda} d\theta = i_{Z_\lambda} i_X d\theta = 0$$

which expresses that the functions W_λ, defined by (14), are first integrals of X. ■

Remark 4.1. By means of (15) it is possible to derive a useful characterization for the infinitesimal generator $\tilde{Z}$ of the one-parameter group $\mathcal{G}$ associated with $\mathcal{F}$ (see the previous section). First of all we observe that the W_λ's given by (14) define a C^∞ function $\tilde{W}$ on $\mathbb{R}\times(\mathbb{R}\times M)$ by $\tilde{W}(\lambda, t, m) = W_\lambda(t, m)$. Next, we consider the 1-form

$$\tilde{\theta} = p_2^* \theta - \tilde{W}\, d\lambda$$

where p_2 denotes the projection operator from $\mathbb{R}\times(\mathbb{R}\times M)$ onto $\mathbb{R}\times M$. Taking into account (15) and the definition of Z_λ, one easily verifies that the following relations hold:

$$i_{\tilde{Z}} d\tilde{\theta} = 0, \qquad \langle \tilde{Z}, d\lambda \rangle = 1 \tag{16}$$

where d now stands for the exterior derivative on $\mathbb{R}\times(\mathbb{R}\times M)$. So it is seen that $\tilde{Z}$ satisfies relations which are formally analogous to those defining a SA vector field.

Remark 4.2. Suppose the given family $\mathcal{F}$ of $d\theta$ symmetries constitutes a one-parameter group, i.e., $F_{\lambda_1} \circ F_{\lambda_2} = F_{\lambda_1+\lambda_2}$ for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. The corresponding family of Z_λ's then reduces to one single vector field, namely, the infinitesimal generator Z of the group. In this case (15) still holds, but one can prove that, in view of the group property of the F_λ's, the expression on the right-hand side of (14) now can be made independent of λ.

Let us now deal with the converse problem: given a first integral of the SA vector field X (which may possibly depend on a parameter), can one derive from it a one-parameter family (or group) of symmetries of $d\theta$.

The answer to this question first requires a closer inspection of the map which assigns to every vector field $Y \in \mathfrak{X}(\mathbb{R}\times M)$ the 1-form $i_Y d\theta$. This map is clearly linear and possesses a one-dimensional kernel, namely, $\mathcal{C}(d\theta)$. In this connection we mention a few properties which, in the special case where θ is the Cartan form corresponding to a Lagrangian system, were considered in some detail by Crampin (1977). The proofs are completely analogous to those given there and will therefore be omitted.

Lemma 4.2. Let $\alpha \in \Omega^1(\mathbb{R} \times M)$ be given; then there exists a vector field $Y \in \mathfrak{X}(\mathbb{R} \times M)$ such that

$$i_Y d\theta = \alpha \tag{17}$$

iff $\langle X, \alpha \rangle = 0$, where X is the SA vector field corresponding to $d\theta$. ■

Corollary 4.3. For a given function $W \in C^\infty(\mathbb{R} \times M)$ there exists a vector field $Y \in \mathfrak{X}(\mathbb{R} \times M)$ such that

$$i_Y d\theta = dW \tag{18}$$

iff W is a first integral of X. ■

Whenever Y is a solution of (18) (for a suitable W) the Lie bracket of Y and X belongs to $\mathcal{C}(d\theta)$, yielding

$$[Y, X] = gX$$

for some $g \in C^\infty(\mathbb{R} \times M)$. This immediately follows by calculating $i_{[Y,X]}d\theta$, using the relation $i_{[Y,X]} = i_Y L_X - L_X i_Y$, and taking into account the definitions of X and Y.

Notice, however, that solutions of (17) or (18) are not uniquely determined. Indeed, suppose $Y \in \mathfrak{X}(\mathbb{R} \times M)$ satisfies one of these relations (for some α or W); then any vector field of the form $Y + fX$, with $f \in C^\infty(\mathbb{R} \times M)$, will also be a solution and one can prove the following interesting result (Crampin, 1977):

Lemma 4.4. Among all solutions of (18) for a given first integral W of X, there is precisely one vector field Y_0 which commutes with X (i.e., $[Y_0, X] = 0$) and for which $\langle Y_0, dt \rangle = 0$. ■

It may be worth mentioning that the previous lemma still holds when the second condition for Y_0 has been replaced, e.g., by $\langle Y_0, dt \rangle = 1$. (This only demands a slight modification in the proof.) As one can easily verify, this particular solution of (18) (which commutes with X and for which the time component is unity) is precisely the SA vector field corresponding to the contact form $d(\theta + W dt)$.

Anyhow, Lemma 4.4. tells us that (18) has a solution Y_0 the (local) flow of which leaves dt invariant and moreover determines a symmetry group of X.

More in line with the present discussion we can prove the following converse to Proposition 4.1.

Proposition 4.5. Given a first integral W of X, possibly depending on λ in a smooth way, (18) defines a smooth family of vector fields to which can be associated a (local) one-parameter family of symmetries of $d\theta$.

Proof. (a) In case W is independent of λ, for every solution Y of (18) we have

$$L_Y d\theta = d(i_Y d\theta) = 0$$

hence Y defines a (local) one-parameter group of symmetries of $d\theta$. (b) For the general case we can define a smooth family of vector fields Z_λ through

$$i_{Z_\lambda} d\theta = dW_\lambda$$

If $\{F_\mu\}$ is the (local) one-parameter family of transformations associated with these Z_λ (see the construction in Section 3), we get in accordance with (9)

$$\frac{d}{d\mu}(F_\mu^* d\theta) = d\left(F_\mu^*\left(i_{Z_\mu} d\theta\right)\right) = F_\mu^* d(dW_\mu) = 0$$

Hence, $F_\mu^* d\theta = F_0^* d\theta = d\theta$, since F_0 is the identity mapping. This completes the proof. ■

Remark 4.3. Until now the assumption that the given contact form is exact was not really essential. If one is not interested in having an explicit formula for W_λ as in (14), the whole analysis could as well have been performed starting from a general contact form (i.e., a closed 2-form of maximal rank).

Summarizing the results of Propositions 4.1 and 4.5, we have seen that a direct link can be established, in both directions, between constants of the motion and symmetries in the sense of Definition 4.1. This notion of $d\theta$ symmetries is, moreover, large enough to allow for general transformations of both "time and space variables." In the next section we will be dealing with a more restricted class of diffeomorphisms, with the purpose of identifying an appropriate notion of canonical transformations in this context, which is close in spirit to the conventional notion and yet helps overcome some of the difficulties indicated in the Introduction. We will come back to the study of symmetries within this restricted class of transformations at the end.

5. CANONICAL TRANSFORMATIONS

In our study of SA vector fields the Hamiltonian model has always served as a guide. Also now, when we are looking for an extension of the classical notion of canonical transformations, we go back to the conventional definition, which says in the first place that a transformation is canonical if every Hamiltonian system is transformed into a Hamiltonian system. Roughly speaking this means that, if we think of the contact form

$$dp_i \wedge dq^i - dH \wedge dt$$

a canonical transformation preserves the "symplectic part" $dp_i \wedge dq^i$, while no specific restriction is imposed on the way the term containing dt is transformed (apart from the fact that canonical transformations are usually considered as being time preserving).

We will now elaborate a similar idea in the case of general SA vector fields on $\mathbb{R} \times M$. For that purpose it seems useful to make some preliminary comments on the structure of exterior forms on the trivial bundle $\mathbb{R} \times M$.

First we notice that each p-form α on $\mathbb{R} \times M$ (with $1 \leq p \leq 2n+1$) admits a splitting into

$$\alpha = \alpha^{(1)} + \alpha^{(2)} \wedge dt \tag{19}$$

where $\alpha^{(2)} \in \Omega^{p-1}(\mathbb{R} \times M)$ and $i_{\partial/\partial t}\alpha^{(1)} = 0$, i.e., $\alpha^{(1)}$ contains no terms in dt.

We then introduce the following equivalence class $[\alpha]$ of p-forms α $(1 \leq p \leq 2n+1)$:

$$[\alpha] = \{\beta \in \Omega^p(\mathbb{R} \times M) | \alpha \wedge dt = \beta \wedge dt\} \tag{20}$$

From (19) and (20) we see that two p-forms α and β will be equivalent iff $\alpha^{(1)} = \beta^{(1)}$.

Finally, for $\alpha \in \Omega^p(\mathbb{R} \times M)$ we put

$$d[\alpha] = \{d\alpha' | \alpha' \in [\alpha]\} \tag{21}$$

with the remark that $d[\alpha] \neq [d\alpha]$.

Henceforth θ will always represent a 1-form on $\mathbb{R} \times M$ such that $d\theta$ is a contact form which determines a SA vector field according to Definition 2.1. In view of (19) θ can be written as

$$\theta = \theta^{(1)} - H\,dt$$

for some $H \in C^\infty(\mathbb{R} \times M)$. ($\theta^{(1)}$ here replaces the canonical 1-form $p_i dq^i$ occurring in the phase space description of Hamiltonian systems.)

It seems to us that the old idea of canonical transformations can be best approached in the present context by requiring for $F \in \mathrm{Diff}(\mathbb{R} \times M)$

$$F_* d[\theta] = d[\theta]$$

Moreover, since canonical transformations should preserve the structure of SA vector fields (with t as a privileged coordinate), some condition must be imposed on the transformation of time. From the local study of SA systems (Sarlet and Cantrijn, 1978a) we already know that time preservation would be a too restrictive condition. In view of (5b), however, it will be sufficient to propose $F_* dt = dt$, allowing in this way translations of time. In fact, one could more generally require $F_* dt = k\, dt$ with k a diffeomorphism on the time axis, but this would merely consist in a rescaling of time.

Combining the previous arguments we finally arrive at the following definition.

Definition 5.1. A map $F \in \mathrm{Diff}(\mathbb{R} \times M)$ will be called a canonical transformation with respect to $[\theta]$ iff

$$\text{(i)} \quad F_* dt = dt \tag{22}$$

$$\text{(ii)} \quad F_* d[\theta] = d[\theta] \tag{23}$$

Lemma 5.1. Let $F \in \mathrm{Diff}(\mathbb{R} \times M)$ be given, with $F_* dt = dt$; then

$$F_* d[\theta] = d[\theta] \Leftrightarrow F_* d\theta = d\theta' \qquad \text{for some } \theta' \in [\theta]$$

Proof. In view of (21) and the definition of the equivalence classes (20), the implication from left to right is obvious. Conversely, suppose $F_* d\theta = d\theta'$ for some $\theta' \in [\theta]$. We then have to show that for all $\theta_1 \in [\theta]$ there exists a $\theta_1' \in [\theta]$ such that

$$F_* d\theta_1 = d\theta_1' \tag{24}$$

By definition, $\theta_1 \in [\theta]$ implies $\theta_1 = \theta + \Psi\, dt$ for some $\Psi \in C^\infty(\mathbb{R} \times M)$. Consequently, taking into account (22), we have

$$\begin{aligned} F_* d\theta_1 &= F_*(d\theta + d\Psi \wedge dt) \\ &= d\theta' + d(\Psi \circ F^{-1}) \wedge dt \\ &= d(\theta' + \Psi \circ F^{-1} dt) \end{aligned}$$

which proves (24) with $\theta_1' = \theta' + (\Psi \circ F^{-1})\, dt$. ■

From this lemma we can immediately derive the following characterization of canonical transformations.

Corollary 5.2. $F \in \mathrm{Diff}(\mathbb{R} \times M)$ is canonical with respect to $[\theta]$ iff $F_* dt = dt$ and

$$F_* d\theta = d(\theta - \phi\, dt) \tag{25}$$

for some $\phi \in C^\infty(\mathbb{R} \times M)$. ■

In this way we see that our definition of canonical transformations is equivalent to the one adopted, e.g., in Abraham and Marsden (1978), except for the possibility of time translations.

Corollary 5.2 also implies that whenever F is canonical we have

$$(F_* d\theta) \wedge dt = d\theta \wedge dt \tag{26}$$

The converse—viz., (26) together with $F_* dt = dt$ imply F canonical—does not hold in general. The full equivalence of both statements is obtained when M is simply connected.

Proposition 5.3. If M is simply connected, then $F \in \mathrm{Diff}(\mathbb{R} \times M)$, with $F_* dt = dt$, is canonical with respect to $[\theta]$ iff (26) holds.

Proof. We have noticed already that F canonical implies (26). Conversely, suppose (26) holds, yielding

$$F_* d\theta = d\theta + \alpha \wedge dt \tag{27}$$

for some $\alpha \in \Omega^1(\mathbb{R} \times M)$, from which it follows with the notations of (19),

$$d\alpha^{(1)} \wedge dt = 0$$

or

$$(d\alpha^{(1)})^{(1)} = 0 \tag{28}$$

Now regarding $\alpha^{(1)}$ as a time-dependent 1-form on M and denoting by d_M the exterior derivative on M, (28) clearly implies $d_M \alpha^{(1)} = 0$. Hence, $\alpha^{(1)} = d_M \beta$, for some time-dependent C^∞ function β, since M is simply connected.

Regarding β as being defined on $\mathbb{R} \times M$, (27) finally can be rewritten as

$$F_* d\theta = d(\theta + \beta\, dt)$$

which completes the proof in view of Corollary 5.2. ■

For a local characterization of canonical transformations we first introduce the following local model. Let $M=U$ be some open subset of $\mathbb{R}^{2n}$ and put

$$\theta = R_i dx^i - H\,dt \tag{29}$$

for some functions $R_i, H \in C^\infty(\mathbb{R}\times U)$. We then have

$$d\theta = \sum_{i<j} C_{ij} dx^i \wedge dx^j - D_i dt \wedge dx^i \tag{30}$$

with C_{ij} and D_i again as in (4).

A transformation F on $\mathbb{R}\times U$ will be denoted by $(t,x)\to(t'(t,x), y(t,x))$ and should be interpreted in the "active" way, i.e., F transforms a point with coordinates (t,x) into a point with coordinates (t',y). Using (22) and (26) it is straightforward to verify that if a C^∞ diffeomorphism F: $\mathbb{R}\times U\to\mathbb{R}\times U$, $(t,x)\to(t'(t,x), y(t,x))$ is canonical with respect to $[\theta]$, the following relations hold:

$$t' = t+\tau \qquad (\tau\in\mathbb{R}) \tag{31a}$$

$$C_{ij}(t',y) \equiv C_{kl}(t'-\tau, x(t',y))\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} \tag{31b}$$

Moreover, according to Proposition 5.3, if U is simply connected each transformation satisfying (31) will be canonical. For $\tau=0$ we recover the class of identity-isotopic transformations studied in Sarlet and Cantrijn (1978a).

So far nothing special has been said about the influence of canonical transformations on SA vector fields. In this connection we first observe that for each $\theta'\in[\theta]$, $d\theta'$ is a contact form on $\mathbb{R}\times M$ which defines a SA vector field according to equations (5).

Proposition 5.4. If $F\in\mathrm{Diff}(\mathbb{R}\times M)$ is canonical with respect to $[\theta]$, then for each $\theta_1\in[\theta]$ there exists a $\theta_2\in[\theta]$ such that F_* converts the SA vector field corresponding to $d\theta_1$ into the SA vector field corresponding to $d\theta_2$.

(The proof is an immediate consequence of the Definitions 2.1 and 5.1 and Corollary 5.2.) ■

In terms of the local representation introduced above, this proposition can be translated as follows. Suppose $F\in\mathrm{Diff}(\mathbb{R}\times U)$ is a canonical transformation with $F(t,x)=(t+\tau, y(t,x))$, then to each $H\in C^\infty(\mathbb{R}\times M)$ there corresponds a function $K \in C^\infty(\mathbb{R} \times M)$ such that the SA system

$$\dot{x}^i = C^{ij}(t, x)\left[\frac{\partial H}{\partial x^j}(t, x) + \frac{\partial R_j}{\partial t}(t, x)\right]$$

is converted into

$$\dot{y}^i = C^{ij}(t+\tau, y)\left[\frac{\partial K}{\partial y^j}(t+\tau, y) + \frac{\partial R_j}{\partial t}(t+\tau, y)\right]$$

The major objection against the concept of identity-isotopic transformations (i.e., time-preserving canonical transformations) was that the evolution of a general SA system could not be described in terms of such transformations. In this respect, the present definition of canonical transformations yields a considerable improvement. We have indeed the following proposition.

Proposition 5.5. The flow of a SA vector field consists of canonical transformations.

Proof. Equations (5a) and (5b) immediately imply

$$L_X d\theta = 0 \quad \text{and} \quad L_X dt = 0$$

Hence, both $d\theta$ and dt are invariant under the flow of X, which proves the above assertion. ∎

In the next two sections, which deal with one-parameter families of canonical transformations and canonical symmetries, it will again be assumed that M is simply connected.

6. ONE-PARAMETER FAMILIES OF CANONICAL TRANSFORMATIONS

Let $\theta \in \Omega^1(\mathbb{R} \times M)$ be given as before and consider a smooth one-parameter family $\mathcal{F} = \{F_\lambda \mid \lambda \in \mathbb{R}\}$ of canonical transformations with respect to $[\theta]$.

By Definition 5.1 we then have for each $\lambda \in \mathbb{R}$

$$(F_\lambda)_* d[\theta] = d[\theta] \tag{32}$$

and

$$(F_\lambda)_* dt = dt \tag{33}$$

From (33) it follows that

$$(F_\lambda)_* t = t + \tau(\lambda) \tag{34}$$

for some $\tau \in C^\infty(\mathbb{R})$, with $\tau(0)=0$ (since F_0 is the identity map). According to Corollary 5.2 there will exist a smooth family of functions $\phi_\lambda \in C^\infty(\mathbb{R} \times M)$ such that (32) may be replaced by

$$(F_\lambda)_* d\theta = d(\theta - \phi_\lambda\, dt)$$

Since, by assumption, $\mathbb{R} \times M$ is simply connected, the previous relation implies the existence of functions $S_\lambda \in C^\infty(\mathbb{R} \times M)$, smoothly depending on λ, such that

$$(F_\lambda)_* \theta = \theta - \phi_\lambda\, dt + dS_\lambda \tag{35}$$

We now seek for a suitable characterization of the vector fields Z_λ, associated with $\mathcal{F}$. From (34) one can immediately deduce

$$\langle Z_\lambda, dt \rangle = \tau'(\lambda) \tag{36}$$

with $\tau' = d\tau/d\lambda$.
After exterior multiplication of (35) with dt and taking into account (34), we get

$$\theta \wedge dt = F_\lambda^* [(\theta + dS_\lambda) \wedge dt]$$

Differentiation of both sides with respect to λ and making use of (9), gives

$$0 = F_\lambda^* \left[d\left(\frac{dS_\lambda}{d\lambda} \right) \wedge dt + i_{Z_\lambda}(d\theta \wedge dt) \right] + dF_\lambda^* \left[i_{Z_\lambda}(\theta \wedge dt + dS_\lambda \wedge dt) \right]$$

After a straightforward calculation we obtain

$$\begin{aligned} 0 = F_\lambda^* &\left[d\left(\frac{dS_\lambda}{d\lambda} + L_{Z_\lambda} S_\lambda + i_{Z_\lambda}\theta \right) \wedge dt \right] \\ &+ F_\lambda^* \left[(i_{Z_\lambda} d\theta) \wedge dt \right] \\ &+ F_\lambda^* (\langle Z_\lambda, dt \rangle d\theta) - dF_\lambda^* (\langle Z_\lambda, dt \rangle \theta) \end{aligned}$$

The last two terms on the right-hand side cancel in view of (36), with $\tau'(\lambda)$ a constant function on $\mathbb{R}\times M$, which finally yields

$$\left(i_{Z_\lambda}d\theta+dW_\lambda\right)\wedge dt=0 \tag{37}$$

where W_λ is again given by (14).

Consequently, there must exist a parameter-dependent function $\Psi_\lambda\in C^\infty(\mathbb{R}\times M)$ such that

$$i_{Z_\lambda}d\theta+dW_\lambda=\Psi_\lambda\, dt \tag{38}$$

Moreover, taking the inner product of (38) with the SA vector field X corresponding to $d\theta$, it is readily seen that

$$L_X W_\lambda=\Psi_\lambda \tag{39}$$

Summarizing we can say that the vector fields Z_λ determined by the given family $\mathcal{F}$ of canonical transformations satisfy the relations (36) and (37) or, equivalently, (36) and (38) with Ψ_λ given by (39). Conversely one can also prove the following proposition:

Proposition 6.1. For each $\tau\in C^\infty(\mathbb{R})$ and each smooth family $\{W_\lambda|\lambda\in\mathbb{R}\}$, with $W_\lambda\in C^\infty(\mathbb{R}\times M)$, the relations (36) and (38) [together with (39)] uniquely define a family of vector fields Z_λ. Moreover, these vector fields locally determine a one-parameter family of canonical transformations.

Proof. Because of (39) and (5b) we immediately have $\langle X,-dW_\lambda+\Psi_\lambda dt\rangle=0$. Hence, according to Lemma 4.2 there will exist a smooth family of vector fields Y_λ for which

$$i_{Y_\lambda}d\theta=-dW_\lambda+\Psi_\lambda\, dt$$

Putting $Z_\lambda=Y_\lambda-(\langle Y_\lambda, dt\rangle-\tau'(\lambda))X$, we define new vector fields, satisfying (36) and (38). They are uniquely determined by these relations.

Let $\{F_\mu\}$ now be a local family of transformations associated with the Z_λ's (see Section 3). We then find, using (9) and taking into account (36) and (38) [with $\tau'(\lambda)$ considered as a constant function on $\mathbb{R}\times M$],

$$\frac{d}{d\mu}F_\mu^* dt=d\left[F_\mu^*\left(i_{Z_\mu}dt\right)\right]=0$$

and

$$\frac{d}{d\mu}F_\mu^*(d\theta\wedge dt)=d\left\{F_\mu^*\left[\left(i_{Z_\mu}d\theta\right)\wedge dt\right]\right\}$$
$$=d\left[F_\mu^*\left(-dW_\mu\wedge dt\right)\right]=0$$

(Hereby it is of course understood that vector fields and differential forms are restricted to a suitable open subset of $\mathbb{R}\times M$.) Consequently, both $F_\mu^* dt$ and $F_\mu^*(d\theta\wedge dt)$ are independent of μ. Since F_0 is the identity map, we find

$$F_\mu^* dt = dt$$

$$F_\mu^*(d\theta\wedge dt) = d\theta\wedge dt$$

Applying Proposition 5.3 (possibly after restricting the domain of definition of the F_μ's) it is seen that the F_μ's are indeed local canonical transformations. ■

From Hamiltonian mechanics we know that each one-parameter family of canonical transformations is generated by a Hamiltonian vector field (see, e.g., Saletan and Cromer, 1971). We now show that a similar property holds within the framework of general SA systems. Suppose $\mathcal{F}=\{F_\lambda \mid \lambda\in\mathbb{R}\}$ is a one-parameter family of canonical transformations with respect to $[\theta]$, such that (32) and (34) hold. The vector fields Z_λ, induced by $\mathcal{F}$ on $\mathbb{R}\times M$, are completely determined by (36) and (38) [together with (39)].

To avoid confusion in notation we will denote the set of reals by $\mathbb{R}'$, when referring to the values of λ, and by $\mathbb{R}''$ when referring to the time axis. Let $\mathcal{G}=\{\tilde{G}_\mu \mid \mu\in\mathbb{R}\}$ be the one-parameter group of transformations defined on $\mathbb{R}'\times\mathbb{R}''\times M$, corresponding to $\mathcal{F}$, with infinitesimal generator $\tilde{Z}\in\mathcal{X}(\mathbb{R}'\times\mathbb{R}''\times M)$ (see Section 3). In view of (34) the diffeomorphisms $\tilde{G}_\mu$ here have the structure [see (7)]:

$$\tilde{G}_\mu(\lambda, t, m) = (\lambda+\mu, t+\tau(\lambda+\mu)-\tau(\lambda), G'_\mu(\lambda, t, m))$$

where G'_μ is a C^∞ map from $\mathbb{R}'\times\mathbb{R}''\times M$ onto M. By means of (36) and (38) we now look for a complete characterization of $\tilde{Z}$. For that purpose we first introduce the following 1-form on $\mathbb{R}'\times\mathbb{R}''\times M$:

$$\tilde{\theta} = p_2^*\theta - W d\lambda \tag{40}$$

with p_2: $\mathbb{R}'\times\mathbb{R}''\times M\to\mathbb{R}''\times M$ the natural projection operator and $W(\lambda, t, m) = W_\lambda(t, m)$. Next, we define a function $\Psi\in C^\infty(\mathbb{R}'\times\mathbb{R}''\times M)$ by $\Psi(\lambda, t, m)=\Psi_\lambda(t, m)$. Using (36) and (38) one easily verifies that the following relations hold:

$$i_{\tilde{Z}} d\tilde{\theta} = \Psi\, dt - \Psi\tau' d\lambda \tag{41a}$$

$$\langle \tilde{Z}, dt\rangle = \tau' \tag{41b}$$

$$\langle \tilde{Z}, d\lambda\rangle = 1 \tag{41c}$$

A first integral of $\tilde{Z}$ is immediately given by $f(\lambda, t, m) = t - \tau(\lambda)$, with $f \in C^\infty(\mathbb{R}' \times \mathbb{R}'' \times M)$. Each $c \in \mathbb{R}''$ is a regular value of f and hence, the corresponding sets $V_c = f^{-1}(c)$ are $(2n+1)$-dimensional submanifolds of $\mathbb{R}' \times \mathbb{R}'' \times M$. The collection of sets $V_c (c \in \mathbb{R}'')$ constitutes a disjoint covering of $\mathbb{R}' \times \mathbb{R}'' \times M$. Moreover, each V_c is diffeomorphic to $\mathbb{R}' \times M$. A diffeomorphism is obtained by taking the restriction to V_c of the projection operator from $\mathbb{R}' \times \mathbb{R}'' \times M$ onto $\mathbb{R}' \times M$. This amounts to saying that we can choose λ as a global coordinate on V_c.

Our purpose is now to consider the reduction of $\tilde{Z}$ onto some V_c. (For a general and rigorous treatment on the reduction of dynamical systems, see, e.g., Marmo et al., 1979a,b.) For a fixed $c \in \mathbb{R}''$ let j_c be the canonical injection from V_c into $\mathbb{R}' \times \mathbb{R}'' \times M$ (i.e., j_c is the identity map of $\mathbb{R}' \times \mathbb{R}'' \times M$ restricted to V_c). From the definition of V_c it then follows that

$$j_c^* dt = \tau' d\lambda \quad \text{and} \quad j_c^* d\lambda = d\lambda \tag{42}$$

We further put

$$j_c^* \tilde{\theta} = \tilde{\theta}_c \tag{43}$$

Since f is a first integral of $\tilde{Z}$, the latter will be tangent to V_c at each point of V_c. Hence, the restriction of $\tilde{Z}$ to V_c defines a true vector field which will be denoted by $\tilde{Z}_c$. Moreover, also as a result of this tangency, it is seen that for each $\alpha \in \Omega^p(\mathbb{R}' \times \mathbb{R}'' \times M)$,

$$j_c^*(i_{\tilde{Z}} \alpha) = i_{\tilde{Z}_c}(j_c^* \alpha) \tag{44}$$

Using (44) and taking into account (42) and (43), the pull-back (by j_c^*) of (41a) onto V_c becomes

$$i_{\tilde{Z}_c} d\tilde{\theta}_c = 0 \tag{45a}$$

whereas, in view of (42), the pull-back of (41b) and (41c) both yield the same relation

$$\langle \tilde{Z}_c, d\lambda \rangle = 1 \tag{45b}$$

Through the diffeomorphism between V_c and $\mathbb{R}' \times M$ mentioned before, we see that $\tilde{Z}_c$ can clearly be thought of as a SA vector field on $\mathbb{R}' \times M$ with respect to $d\tilde{\theta}_c$.

Replacing c everywhere by t, we can summarize the preceding as follows:

> *Theorem 6.2.* Each one-parameter family of canonical transformations $\{F_\lambda | \lambda \in \mathbb{R}\}$ is generated by a parameter-dependent SA vector field $\tilde{Z}_t$. ■

To close this section we illustrate Theorem 6.2 on the local model introduced in the previous section [with $M = U \subset \mathbb{R}^{2n}$, θ and $d\theta$ being given, respectively, by (29) and (30)].

Suppose $\{F_\lambda | \lambda \in \mathbb{R}\}$ is a one-parameter family of canonical transformations with respect to $[\theta]$, where $F_\lambda \in \mathrm{Diff}(\mathbb{R} \times U)$ is of the form

$$F_\lambda(t, x) = (t + \tau(\lambda), y(\lambda, t, x))$$

From its definition in equations (40) and (43), and taking into account the relations (42), it is clear that the 1-form $\tilde{\theta}_t$ here will read

$$\tilde{\theta}_t(\lambda, y) = R_i(t+\tau(\lambda), y)\,dy^i - \big[\tau'(\lambda) H(t+\tau(\lambda), y) + W(\lambda, t+\tau(\lambda), y)\big]\,d\lambda \tag{46}$$

The scope of Theorem 6.2 then means that the functions $y(\lambda, t, x)$ are the solutions of the SA system

$$C_{ij}(t+\tau(\lambda), y)\frac{dy^j}{d\lambda} - \left\{\frac{\partial W(\lambda, t+\tau(\lambda), y)}{\partial y^i} + \frac{\partial}{\partial y^i}\big[\tau'(\lambda) H(t+\tau(\lambda), y)\big] + \frac{\partial R_i(t+\tau(\lambda), y)}{\partial \lambda}\right\} = 0 \tag{47}$$

with t as parameter and initial value $y(0, t, x) = x$.

7. CANONICAL SYMMETRIES

In this section we study symmetries of a SA system in terms of canonical transformations.

Definition 7.1. $F \in \mathrm{Diff}(\mathbb{R} \times M)$ is called a canonical symmetry iff F is a symmetry of $d\theta$ and moreover satisfies $F_* dt = dt$.

Obviously, each canonical symmetry is in particular a canonical transformation.

In Section 4 it was seen that a general symmetry of $d\theta$ need not be a symmetry of the SA vector field X (in the sense of Definition 4.2). When dealing with canonical transformations, however, both concepts of symmetry are equivalent. Indeed, we have the following proposition.

Proposition 7.1. A canonical transformation F is a symmetry of $d\theta$ iff it is a symmetry of the corresponding SA vector field.

Proof. If F is a symmetry of $d\theta$ we already know that $F_*X=fX$ for some $f\in C^\infty(\mathbb{R}\times M)$ (see Section 4). From (5b) we then obtain

$$f\langle X, F_*dt\rangle=1$$

which, expressing that F is canonical, yields $f=1$. Hence, F is a symmetry of X. Conversely, if $F_*X=X$ it follows from (5a) that

$$i_X F_* d\theta=0 \tag{48}$$

F being a canonical transformation we also have $F_*d\theta=d\theta-d\phi\wedge dt$, for some $\phi\in C^\infty(\mathbb{R}\times M)$ (see Corollary 5.2). Substitution into (48) gives, using (5a) and (5b),

$$0=i_X(d\phi\wedge dt)=(i_X d\phi)dt-d\phi$$

Consequently, $d\phi\wedge dt=0$ and so $F_*d\theta=d\theta$, which completes the proof. ■

All results of Section 4 especially hold for canonical symmetries. In particular, each one-parameter family of canonical symmetries $\{F_\lambda | \lambda\in\mathbb{R}\}$, with $(F_\lambda)_*t=t+\tau(\lambda)$, defines a parameter-dependent vector field Z_λ for which

$$i_{Z_\lambda}d\theta=-dW_\lambda \quad \text{and} \quad \langle Z_\lambda, dt\rangle=\tau'(\lambda)$$

The functions W_λ hereby determine a parameter-dependent constant of the motion of the SA vector field corresponding to $d\theta$. Conversely, combining Lemma 4.4 and Proposition 4.5, it follows that each first integral of the SA system (5) can be (locally) generated by a one-parameter group (or family) of canonical symmetries. (This is also implicitly contained in Proposition 6.1, if W_λ is taken to be a first integral of X and hence, $\Psi_\lambda=0$.) In the next section we illustrate some of the previous results on a local example.

8. EXAMPLE: EMDEN EQUATION

Consider the second-order differential equation

$$\ddot{x}+\frac{2}{t}\dot{x}+x^5=0 \tag{49}$$

which was introduced in astrophysics by Emden. Putting $y=\dot{x}$ one can pass to the equivalent first-order system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^5 - 2y/t \end{aligned} \tag{50}$$

As can be readily verified, these equations define a SA system corresponding to

$$d\theta = -t^2 dx \wedge dy - (t^2x^5 + 2ty)\, dx \wedge dt - t^2 y\, dy \wedge dt \tag{51}$$

where

$$\theta = t^2 y\, dx - \tfrac{1}{2}t^2\left(y^2 + \tfrac{1}{3}x^6\right) dt$$

In the notations of our local description [(29) and (30)], we have $R_1 = t^2 y$, $R_2 = 0$, $H = \frac{1}{2}t^2(y^2 + \frac{1}{3}x^6)$. The SA vector field is here given by

$$X = y\frac{\partial}{\partial x} + \left(-x^5 - \frac{2y}{t}\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$$

In the present case M can, e.g., be identified with $\mathbb{R}^2$. However, in order that $d\theta$ would be of constant rank 2, one should restrict the time axis to some open interval not containing the origin. Obviously, restrictions of that kind do not affect the validity of the previous results.

From a simple inspection of $d\theta$, one can detect the following one-parameter family of $d\theta$ symmetries:

$$x = \bar{x}e^{\lambda}, \qquad y = \bar{y}e^{3\lambda}, \qquad t = \bar{t}e^{-2\lambda} \tag{52}$$

These transformations moreover constitute a group, the infinitesimal generator of which is given by

$$Y = x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2t\frac{\partial}{\partial t}$$

Calculating $i_Y d\theta$ we obtain

$$i_Y d\theta = -(2t^3x^5 + t^2y)\, dx - (t^2x + 2t^3y)\, dy - (t^2x^6 + 2tyx + 3t^2y^2)\, dt$$

which is indeed minus the total differential of the function

$$W = t^3\left(y^2 + \tfrac{1}{3}x^6\right) + t^2xy \tag{53}$$

Hence, we conclude that W is a first integral of the SA system (50), recovering in this way a well-known result (see, e.g., Logan, 1977, p. 52).

Whereas the transformations (52) are general symmetries of $d\theta$, it follows from the discussion at the end of the previous section that the same constant of motion W also corresponds to a one-parameter group of canonical symmetries. An infinitesimal generator of such a group is obtained by adding to the generator Y of (52) a multiple of X, say fX, such that $\langle Y+fX, dt\rangle$ becomes a constant. Let us, e.g., choose $f=1-\langle Y, dt\rangle=1+2t$. The vector field $Z=Y+(1+2t)X$, which is explicitly given by

$$Z=\left[x+(1+2t)y\right]\frac{\partial}{\partial x}-\left[y\left(1+\frac{2}{t}\right)+x^5(1+2t)\right]\frac{\partial}{\partial y}+\frac{\partial}{\partial t}$$

clearly satisfies $i_Z d\theta=-dW$, with W given by (53), and the flow of Z will consist of canonical symmetries of $d\theta$. However, as can be seen from the structure of Z, the construction of this flow, i.e., the integration of the differential equations corresponding to Z, becomes extremely difficult.

9. DISCUSSION

In analogy with the definition of time-dependent Hamiltonian vector fields (within the Cartan description) we have introduced a SA vector field on a contact manifold $(\mathbb{R}\times M, d\theta)$ as a characteristic vector field of $d\theta$, for which the "time" component is unity.

The study of symmetries of SA systems revealed that, although symmetries of $d\theta$ do not constitute the most general transformations mapping integral curves of the SA vector field into integral curves, they are particularly useful. First of all there is a very simple link between symmetries and conservation laws, and secondly the framework is large enough to allow both space and time transformations. We have also introduced a generalized notion of canonical transformations which should not necessarily be strictly time preserving. As a matter of fact, the possibility for time translations was crucial for the result that the flow of a SA vector field is canonical.

We further showed that a correspondence in both directions between symmetries and constants of the motion also exists within the restricted class of canonical transformations. This means for instance that, on theoretical grounds, no constants of the motion are lost if one would only want to deal with canonical symmetries of SA systems. However, as we have learned from a simple example like the Emden equation, the detection of canonical symmetries might become extremely difficult in practice, even in those cases where a close inspection of $d\theta$ suffices to produce a general $d\theta$ symmetry.

Herewith, we have touched an important point in relation to the range of practical applicability of the theoretical results. Indeed, these results show us, for example, that once a symmetry is found one can immediately compute a first integral. They do not, however, give any indication about how such a symmetry could be found. This shows the need for a deeper study concerning the practical usefulness of the formulas obtained in this paper.

In a forthcoming paper we therefore plan to study the problem of how to find useful partial differential equations for the detection of symmetries, together with other questions, such as: what is the relation with, e.g., the Killing equations known in the context of Noether's theorem (see, e.g., Vujanovic, 1970; Djukic, 1973), and how can one exploit the knowledge that a canonical symmetry is generated by a SA vector field?

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